SUMS OF SQUARES FROM ELLIPTIC PFAFFIANS

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Abstract. We give a new proof of Milne's formulas for the number of representations of an integer as a sum of $4m^2$ and $4m(m+1)$ squares. The proof is based on explicit evaluation of pfaffians with elliptic function entries, and relates Milne's formulas to Schur Q-polynomials and to correlation functions for continuous dual Hahn polynomials. We also state a new formula for $2m^2$ squares.

1. Introduction

One of the classical problems of number theory is to count the number of representations of a positive integer n as a sum of k squares. We will denote this number by $\square_k(n)$, where, as is customary, representations

$$n = x_1^2 + \cdots + x_k^2$$

that may be obtained from each other by permuting the x_i, or replacing some x_i by $-x_i$, are counted as different.

The most fundamental results are Gauss' two squares and Jacobi's four and eight squares formulas:

$$\square_2(n) = 4 \sum_{d|n,\, d \text{ odd}} (-1)^{\frac{1}{2}(d-1)}, \tag{1.1a}$$

$$\square_4(n) = 8 \sum_{d|n,\, 4\nmid d} d, \tag{1.1b}$$

$$\square_8(n) = 16 \sum_{d|n} (-1)^{n+d} d^3, \tag{1.1c}$$

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where the sums run over positive divisors. These may be compared with Legendre's formulas for sums of triangles:

$$\triangle_2(n) = \sum_{d|4n+1} (-1)^{\frac{1}{2}(d-1)}, \tag{1.2a}$$

$$\triangle_4(n) = \sum_{d|2n+1} d, \tag{1.2b}$$

$$\triangle_8(n) = \sum_{d|n+1,\,(n+1)/d \text{ odd}} d^3. \tag{1.2c}$$

It is known that, in general, $\square_{2k}(n)$ can be written as the sum of two terms, the first being an elementary divisor sum and the second the n-th Fourier coefficient of a cusp form. Moreover, the second term vanishes only for $k \leq 4$, which indicates that there is no very simple extension of (1.1) to more than 8 squares.

A novel approach, motivated by *affine superalgebras*, was initiated by Kac and Wakimoto [KW], who used *denominator formulas* for such algebras to derive several new infinite families of identities, both for squares and triangles. A particularly interesting case is the "queer" series of affine superalgebras $Q(m)$, for which the denominator formula was merely conjectured. For $Q(2m-1)$ and $Q(2m)$, respectively, it implies the triangular number identities

$$\triangle_{4m^2}(n) = \frac{1}{4^{m(m-1)}\prod_{j=1}^{2m-1} j!} \sum_{\substack{k_1l_1+\dots+k_ml_m=2n+m^2 \\ k_1>k_2>\dots>k_m \\ k_i \text{ and } l_i \text{ odd positive}}} \prod_{i=1}^m k_i \prod_{1\leq i<j\leq m} (k_i^2-k_j^2)^2, \tag{1.3a}$$

$$\triangle_{4m(m+1)}(n) = \frac{2^m}{\prod_{j=1}^{2m} j!} \sum_{\substack{k_1l_1+\dots+k_ml_m=n+\frac{1}{2}m(m+1) \\ k_1>k_2>\dots>k_m \\ k_i \text{ positive},\ l_i \text{ odd positive}}} \prod_{i=1}^m k_i^3 \prod_{1\leq i<j\leq m} (k_i^2-k_j^2)^2. \tag{1.3b}$$

The case $m=1$ gives Legendre's formulas for four and eight triangles. The sums of squares formulas contained in [KW] are different from those discussed in the present paper.

The identities (1.3) were first proved by Milne [M3], see also the research announcements [M1, M2], using an approach different from Kac and Wakimoto. Independently, the more general denominator formula was proved by Zagier [Z]. Getz and Mahlburg [GM] showed that it contains further triangular number identities, such as

$$\triangle_{2m}(n) = \sum_{\substack{k_1l_1+\dots+k_ml_m=4mn+m^2 \\ k_i \text{ and } l_i \text{ odd positive} \\ k_i\equiv\pm(2i-1) \mod 4m}} (-1)^{|\{i;\, k_i\equiv 1-2i \ (4m)\}|}, \tag{1.4}$$

which reduces to (1.2a) for $m=1$.

In [R1], we made the observation that the denominator formulas for queer affine superalgebras can be written as *pfaffian evaluations*, for matrices with elliptic function entries. Moreover, we showed that they follow from a classical determinant evaluation due to Frobenius [F]. We also attempted a complete study of the implied triangular number identities, finding formulas for $4m^2/d$ triangles, when $d \mid 2m$, and $4m(m+1)/d$ triangles, when $d \mid 2m$ or $d \mid 2m+2$. As an example, letting $d=2$ in [R1, Eq. (4.4b)] gives a $2m^2$ triangles identity, which can be written as

$$\triangle_{2m^2}(n) = \frac{(-1)^{\frac{1}{2}m(m-1)}}{4^{m(m-1)}\prod_{j=1}^{m-1}(j!)^2} \sum_{\substack{k_1l_1+\dots+k_ml_m=4n+m^2\\ k_1>k_2>\dots>k_m\\ k_i \text{ and } l_i \text{ odd positive}}} \prod_{i=1}^{m}(-1)^{\frac{1}{2}(k_i-1)} \times \prod_{1\le i<j\le m}\Big((-1)^{\frac{1}{2}(k_j-1)}k_j - (-1)^{\frac{1}{2}(k_i-1)}k_i\Big)^2. \quad (1.5)$$

Milne's proof of (1.3) also uses elliptic functions, but is based on continued fractions and Hankel determinants, rather than on pfaffians. With similar methods he obtained new formulas for sums of $4m^2$ and $4m(m+1)$ squares, see Corollary 4.5 below. As an example, Milne's 16 squares formula can be written [M3, Corollary 8.1]

$$\square_{16}(n) = \frac{2^5}{3}\sum_{kl=n,\,k,l\ge 1}(-1)^{(k-1)(l-1)}k(1+k^2+k^4) + \frac{2^8}{3}\sum_{\substack{k_1l_1+k_2l_2=n\\ k_1>k_2\ge 1,\,l_1,l_2\ge 1}}(-1)^{(k_1-1)(l_1-1)+(k_2-1)(l_2-1)}k_1k_2(k_1^2-k_2^2)^2.$$

For a readable introduction to Milne's work, we refer to the survey [CK].

Ono [On] derived seemingly different formulas for $4m^2$ and $4m(m+1)$ squares from (1.3), using the fact that the generating functions for squares and triangles are related by a modular transformation, cf. (2.9) below. Moreover, in the introduction he indicates an alternative proof, also using modular forms but without relying on (1.3). In the Appendix, we will show that Ono's formulas are in fact equivalent to Milne's, although his proof is different. Another modular forms proof of Milne's formulas was recently given by Long and Yang [LY]. This should be related to the alternative proof indicated by Ono.

The purpose of the present work is to extend the analysis of [R1] from triangles to squares, deriving Milne's sums of squares formulas from elliptic pfaffian evaluations. These evaluations follow from Frobenius' classical determinant, and may be viewed as modular duals of those used in [R1]. Thus, just as in Ono's proof square number identities arise as modular duals of triangular number identities, although the details are different.

To be precise, we give two, closely related, derivations of Milne's formulas. Applying two different modified Laurent series expansions to the matrix elements of our pfaffians, we obtain two different analogues of the Kac–Wakimoto denominator formula, Theorem 3.5 and Theorem 5.5. Theorem 3.5 looks very similar to the Kac–Wakimoto formula, though convergent Lambert series are replaced by Abel means of divergent series. Theorem 5.5 involves convergent series, but Schur polynomials are replaced by *Schur Q-polynomials*. To be precise, classical Schur Q-polynomials are labelled by positive integer partitions; here, we need an extension to the case when some labels are negative.

Theorems 3.5 and 5.5 involve a number of free variables. Specializing all variables to 1 gives formulas for $4m^2$ and $4m(m+1)$ squares. The resulting formulas are equivalent, though to see that is far from obvious. Starting with Theorem 3.5, one readily obtains the Hankel determinant form of Milne's identities, Corollary 3.6. Applying an identity relating Hankel determinants to *correlation functions* of orthogonal polynomial ensembles, we deduce explicit sums of squares formulas involving correlation functions for *continuous dual Hahn polynomials*, see Corollary 4.4. Expanding the correlation functions into Schur polynomials yields the Schur function form of Milne's identities, Corollary 4.5.

On the other hand, specializing all variables in Theorem 5.5 to 1 yields sums of squares formulas involving similarly specialized Schur Q-polynomials, Corollary 5.7. Such quantities, and more generally Schur Q-polynomials specialized to a geometric progression, are studied in [R2]. It follows from that work that the Schur Q-polynomials appearing in Corollary 5.7 agree with the correlation functions of Corollary 4.4, and consequently that Corollary 5.7 is equivalent to Milne's identities.

It is natural to try to generalize Milne's formulas to other numbers of squares, similarly as was done for triangular numbers in [GM, R1]. In principle, it should be possible to obtain formulas for $4m^2/d$ squares, when $d \mid 2m$ and $4m(m+1)/d$ squares, when $d \mid 2m$ or $d \mid 2(m+1)$, from our elliptic pfaffian evaluations. However, in practice the computations are difficult to handle, and the resulting identities seem very complicated to state. As an indication of what the results may look like, we state without proof a new formula for $2m^2$ squares in Theorem 6.1. In particular, this embeds the two squares formula (1.1a) in an infinite family, similarly as Milne's identities do for four and eight squares.

The reader may question the merits of our new proof of Milne's formulas, compared to the simple modular forms proofs of Ono and of Long and Yang. First of all, in our approach (similarly as in Milne's original proof) the identities are *derived* from classical results. Although modular forms are a powerful tool for verifying Milne's formulas, they seem less useful for deriving the results from scratch. Second, our proof explains the coefficients in Milne's formulas (i.e. the double sums over λ and μ in Corollary 4.5), by relating them to Schur Q-polynomials and to correlation functions for continuous dual Hahn polynomials. Third, we obtain Milne's formulas as a special case of the more general Theorems 3.5 and 5.5, which

might be of independent interest, plausibly in the theory of superalgebras. In conclusion, we feel that the three known approaches to Milne's identities (Milne's original proof using continued fractions and Hankel determinants, the modular forms proofs of Ono and of Long and Yang and the present work using pfaffian evaluations) each add something to the understanding of these deep and beautiful results.

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2. Preliminaries

2.1. **Pfaffians.** As in [R1], we define the pfaffian of a skew-symmetric matrix as

$$\operatorname*{pfaff}_{1\leq i,j\leq m}(a_{ij})=\frac{1}{2^M M!}\sum_{\sigma\in S_m}\operatorname{sgn}(\sigma)\prod_{i=1}^{M}a_{\sigma(2i-1),\sigma(2i)},$$

where M is the integral part of $m/2$. This definition is standard when m is even, but not when m is odd. In even dimension,

$$\det_{1\leq i,j\leq 2m}(a_{ij})=\Big(\operatorname*{pfaff}_{1\leq i,j\leq 2m}(a_{ij})\Big)^2. \tag{2.1}$$

The cases of odd and even dimension are related by

$$\operatorname*{pfaff}_{1\leq i,j\leq 2m+1}(a_{ij})=\operatorname*{pfaff}_{1\leq i,j\leq 2m+2}\begin{pmatrix} & & & 1\\ & a_{ij} & & \vdots\\ & & & 1\\ -1 & \cdots & -1 & 0\end{pmatrix}.$$

We note in passing the pfaffian evaluation

$$\operatorname*{pfaff}_{1\leq i,j\leq m}\left(\frac{x_i-x_j}{x_i+x_j}\right)=\prod_{1\leq i<j\leq m}\frac{x_i-x_j}{x_i+x_j}. \tag{2.2}$$

When m is even, this is a classical identity of Schur [Sc]. The case of odd m then follows by letting $x_m=0$.

2.2. **Theta functions.** Throughout, q will be a fixed number with $0<q<1$. We will use the notation

$$(a)_\infty=(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k),$$

$$(a_1,\dots,a_m)_\infty=(a_1,\dots,a_m;q)_\infty=(a_1;q)_\infty\cdots(a_m;q)_\infty.$$

When the base is suppressed from the notation, it is always taken to equal our fixed number q.

We introduce the theta function

$$\theta(x)=\theta(x;q)=(x,q/x;q)_\infty.$$

It satisfies

$$\theta(x^{-1}) = \theta(qx) = -x^{-1}\theta(x) \tag{2.3}$$

and the modular transformation

$$\begin{aligned}\theta(e^{2\pi i x}; e^{-2\pi/h})\\ &= -i\sqrt{h}\, e^{-\frac{\pi}{4}(h-h^{-1})} \frac{(e^{-2\pi h}; e^{-2\pi h})_\infty}{(e^{-2\pi/h}; e^{-2\pi/h})_\infty}\, e^{\pi x(i+h(1-x))}\, \theta(e^{-2\pi h x}; e^{-2\pi h}).\end{aligned} \tag{2.4}$$

The Laurent expansion of θ is given by Jacobi's triple product identity

$$(q)_\infty\, \theta(x) = \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} x^k. \tag{2.5}$$

We also mention the Laurent expansion

$$\frac{(q)_\infty^2 \theta(ax)}{\theta(a)\theta(x)} = \sum_{k=-\infty}^{\infty} \frac{x^k}{1-aq^k}, \qquad q < |x| < 1, \tag{2.6}$$

which is a special case of Ramanujan's ${}_1\psi_1$ sum [GR, Eq. (5.2.1)], together with its limit case

$$x\,\frac{\theta'(x)}{\theta(x)} = -\sum_{k\neq 0} \frac{x^k}{1-q^k}, \qquad q < |x| < 1. \tag{2.7}$$

2.3. **Generating functions.** The triple product identity (2.5) implies explicit formulas for the generating functions for squares and triangles. Indeed, let

$$\Box(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2},$$

$$\triangle(q) = \frac{1}{2}\sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n(n+1)} = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)},$$

so that, according to the standard conventions that we use,

$$\Box(q)^k = \sum_{n=0}^{\infty} \Box_k(n) q^n,$$

$$\triangle(q)^k = \sum_{n=0}^{\infty} \triangle_k(n) q^n.$$

Then, by (2.5),

$$\Box(q) = (q^2;q^2)_\infty\, \theta(-q;q^2) = (q^2,-q,-q;q^2)_\infty = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{2.8}$$

$$\triangle(q) = \frac{1}{2}\,(q;q)_\infty\, \theta(-q;q) = (q,-q,-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

Note that the special case $x = 1/2 + i/h$ of (2.4) gives

$$\triangle(e^{-2\pi/h}) = \frac{\sqrt{h}}{2}\, e^{\pi/4h}\, \square(-e^{-\pi h}). \tag{2.9}$$

We recall the Lambert series versions of (1.1), that is,

$$\square(q)^2 = 1 + 4\sum_{k=1}^{\infty} \frac{q^k}{1+q^{2k}}, \tag{2.10a}$$

$$\square(q)^4 = 1 + 8\sum_{k=1}^{\infty} \frac{kq^k}{1+(-q)^k}, \tag{2.10b}$$

$$\square(q)^8 = 1 + 16\sum_{k=1}^{\infty} \frac{k^3q^k}{1-(-q)^k}. \tag{2.10c}$$

Expanding the denominators as geometric series leads immediately to (1.1a) and (1.1c). In the case of four squares, one obtains

$$8\sum_{d|n}(-1)^{(d-1)(\frac{n}{d}-1)}d,$$

which is equivalent to (1.1b); cf. the case $k = 1$ of (A.1) below.

In §6, we need two further identities of Jacobi. First, we have the Lambert series expansion

$$q\triangle(q^2)^4\,\square(q)^2 = \sum_{k=1}^{\infty} \frac{k^2q^k}{1+q^{2k}}. \tag{2.11}$$

Second, comparing (1.1b) and (1.2b) gives

$$8\triangle_4(n) = \square_4(2n+1),$$

which can equivalently be written

$$16q\triangle(q^2)^4 = \square(q)^4 - \square(-q)^4. \tag{2.12}$$

2.4. **Sums of triangular numbers.** For comparison, we briefly recall how the triangular number identities (1.3) follow from the pfaffian evaluations

$$\operatorname*{pfaff}_{1\le i,j\le 2m}\left(\frac{\theta(x_j/x_i)}{x_j\theta(\sqrt{q}x_j/x_i)}\right) = q^{\frac14 m(m-1)}\prod_{i=1}^{2m}x_i^{m-i}\prod_{1\le i<j\le 2m}\frac{\theta(x_j/x_i)}{\theta(\sqrt{q}x_j/x_i)}, \tag{2.13a}$$

$$\begin{aligned}\operatorname*{pfaff}_{1\le i,j\le 2m+1}&\left(\frac{x_i\theta'(\sqrt{q}x_i/x_j)}{x_j\theta(\sqrt{q}x_i/x_j)}\right)\\ &= q^{\frac14 m(m-1)}\frac{(q)_\infty^{2m}}{(\sqrt{q})_\infty^{2m}}\prod_{i=1}^{2m+1}x_i^{m+1-i}\prod_{1\le i<j\le 2m+1}\frac{\theta(x_j/x_i)}{\theta(\sqrt{q}x_j/x_i)}.\end{aligned} \tag{2.13b}$$

For further details we refer to [R1], though the essential points are contained already in [Z].

It will be convenient to introduce the functions

$$S_\mu(x_1,\dots,x_m)=\frac{\det_{1\leq i,j\leq m}(x_j^{\mu_i})}{\prod_{1\leq i<j\leq m}(x_i-x_j)},\qquad \mu\in\mathbb{Z}^m, \tag{2.14}$$

which are essentially Schur polynomials. Indeed, since S_μ is anti-symmetric in the variables μ_i we may assume $\mu_1>\dots>\mu_m$. Moreover, since

$$S_{(\mu_1+a,\dots,\mu_m+a)}(x_1,\dots,x_m)=x_1^a\cdots x_m^a\,S_{(\mu_1,\dots,\mu_m)}(x_1,\dots,x_m) \tag{2.15}$$

we may also assume $\mu_m\geq 0$. Then, writing $\mu_i=\lambda_i+m-i$, so that $\lambda_1\geq\dots\geq\lambda_m\geq 0$, S_μ equals the Schur polynomial s_λ.

Returning to the task at hand, we expand each matrix element in (2.13) as a Laurent series in the annulus $\sqrt{q}<|x_j/x_i|<1/\sqrt{q}$. Applying (2.6) and (2.7) one arrives after some elementary manipulation at the multivariable Lambert series

$$\begin{split}\frac{(q)_\infty^{2m}}{(\sqrt{q})_\infty^{2m}}\prod_{i=1}^{2m}\frac{1}{x_i^m}\prod_{1\leq i<j\leq 2m}\frac{(qx_j/x_i,qx_i/x_j)_\infty}{(\sqrt{q}x_j/x_i,\sqrt{q}x_i/x_j)_\infty}=q^{-\frac14 m(m-1)}\\ \times\sum_{k_1>\dots>k_m\geq 0}\prod_{i=1}^m\frac{q^{\frac12 k_i}}{1-q^{k_i+\frac12}}\,S_{(k_1,\dots,k_m,-k_m-1,\dots,-k_1-1)}(x_1,\dots,x_{2m}),\end{split} \tag{2.16a}$$

$$\begin{split}\frac{(q)_\infty^{2m}}{(\sqrt{q})_\infty^{2m}}\prod_{i=1}^{2m+1}\frac{1}{x_i^m}\prod_{1\leq i<j\leq 2m+1}\frac{(qx_j/x_i,qx_i/x_j)_\infty}{(\sqrt{q}x_j/x_i,\sqrt{q}x_i/x_j)_\infty}\\ =q^{-\frac14 m(m+1)}\sum_{k_1>\dots>k_m\geq 1}\prod_{i=1}^m\frac{q^{\frac12 k_i}}{1-q^{k_i}}\,S_{(k_1,\dots,k_m,0,-k_m,\dots,-k_1)}(x_1,\dots,x_{2m+1}),\end{split} \tag{2.16b}$$

which are equivalent to the denominator formula for $Q(2m-1)$ and $Q(2m)$, respectively.

Specializing $x_i\equiv 1$, the left-hand sides of (2.16) reduce to $\triangle(\sqrt{q})^{4m^2}$ and $\triangle(\sqrt{q})^{4m(m+1)}$, respectively. On the right, applying the classical formula

$$S_\mu(1^m)=\prod_{1\leq i<j\leq m}\frac{\mu_i-\mu_j}{j-i} \tag{2.17}$$

and expanding the denominators as geometric series leads after simplifications to (1.3).

More generally, one may let $x_i=\omega^{i-1}$, with ω a suitable root of unity. This leads to more general triangular number identities such as (1.4) and (1.5), see [R1].

2.5. **Hankel determinants.** We recall the following classical result [I, Corollary 2.1.3].

Lemma 2.1. *Let*
$$\mu(f)=\int f(x)\,d\mu(x)$$
be a linear functional defined on polynomials, let $c_k=\mu(x^k)$ be its moments, and let
$$\Delta(x)=\prod_{1\leq i<j\leq m}(x_j-x_i)=\det_{1\leq i,j\leq m}\left(x_i^{j-1}\right)$$
denote the Vandermonde determinant. Then,
$$\det_{1\leq i,j\leq m}(c_{i+j-2})=\frac{1}{m!}\int\Delta(x_1,\dots,x_m)^2\,d\mu(x_1)\cdots d\mu(x_m).$$

Proof. Clearly,
$$\int\Delta(x_1,\dots,x_m)^2\,d\mu(x_1)\cdots d\mu(x_m)=\sum_{\sigma,\tau\in S_m}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\prod_{i=1}^m c_{\sigma(i)+\tau(i)-2}.$$
Replacing σ by $\sigma\tau$, this may indeed be written
$$\sum_{\sigma,\tau\in S_m}\operatorname{sgn}(\sigma)\prod_{i=1}^m c_{i+\sigma(i)-2}=m!\det_{1\leq i,j\leq m}(c_{i+j-2}).$$
□

Equivalently, defining the coefficients $C(k_1,\dots,k_m)$ by
$$\prod_{1\leq i<j\leq m}(x_j-x_i)^2=\sum_{k_1,\dots,k_m}C(k_1,\dots,k_m)\prod_{i=1}^m x_i^{k_i},$$
we have, for arbitrary scalars c_k,
$$\sum_{k_1,\dots,k_m}C(k_1,\dots,k_m)\prod_{i=1}^m c_{k_i}=m!\det_{1\leq i,j\leq m}(c_{i+j-2}).$$
This version of Lemma 2.1 will be useful for our discussion of Ono's identities in the Appendix.

If p_k and q_k are arbitrary monic polynomials of degree k then, by linearity,
$$\det_{1\leq i,j\leq m}\left(\mu(p_{i-1}q_{j-1})\right)=\det_{1\leq i,j\leq m}\left(\mu(x^{i+j-2})\right).$$
In particular, if μ is a positive functional we may choose $p_k=q_k$ as the corresponding monic orthogonal polynomials. Then, Lemma 2.1 gives
$$\frac{1}{m!}\int\Delta(x_1,\dots,x_m)^2\,d\mu(x_1)\cdots d\mu(x_m)=\prod_{i=1}^m\|p_{i-1}\|^2. \tag{2.18}$$

2.6. **Tangent numbers.** We will need the following evaluation of Abel means of alternating power sums in terms of tangent numbers, see [T, Theorem 2.5]. It is equivalent to the classical evaluation of Riemann's zeta function at the negative integers, though for completeness we provide a self-contained proof. We find it convenient to define t_k by

$$\tan\frac{x}{2}=\sum_{k=1}^{\infty}t_k\frac{x^{2k-1}}{(2k-1)!}. \tag{2.19}$$

Equivalently, in standard notation for tangent and Bernoulli numbers,

$$t_k=2^{1-2k}T_k=\frac{(4^k-1)|B_{2k}|}{k}.$$

We will denote Abel means of possibly divergent series by

$$\sum_{k\in\Lambda}{}' c_k=\lim_{t\to 1^-}\sum_{k\in\Lambda}t^{|k|}c_k, \qquad \Lambda\subseteq\mathbb{Z}. \tag{2.20}$$

Lemma 2.2. *One has*

$$\sum_{k=1}^{\infty}{}'(-1)^{k+1}k^m=\begin{cases}1/4, & m=0,\\ 0, & m=2,4,6,\dots,\\ (-1)^{n+1}t_n/2, & m=2n-1.\end{cases}$$

Proof. We assume that $m>0$. Since

$$\frac{1-x}{1+x}=1+2\sum_{k=1}^{\infty}(-1)^kx^k, \qquad |x|<1, \tag{2.21}$$

we may write

$$\begin{aligned}2\sum_{k=1}^{\infty}{}'(-1)^{k+1}k^m&=\left(x\frac{d}{dx}\right)^m\bigg|_{x=1}\frac{1-x}{1+x}=\left(\frac{d}{dt}\right)^m\bigg|_{t=0}\frac{1-e^t}{1+e^t}\\&=\left(\frac{d}{dt}\right)^m\bigg|_{t=0}\left(i\tan\frac{t}{2i}\right),\end{aligned}$$

which implies the desired result. □

We introduce the moment functionals

$$\mu_\varepsilon(f)=2\sum_{k=1}^{\infty}{}'(-1)^{k+1+\varepsilon}k^{1+2\varepsilon}f(-k^2), \qquad \varepsilon=0,1. \tag{2.22}$$

Corollary 2.3. *The moments of μ_0 and μ_1 are given by*

$$\mu_0(x^k)=\mu_1(x^{k-1})=t_{k+1}. \tag{2.23}$$

In Lemma 4.1, we give alternative expressions for the functionals μ_ε. As a consequence, we shall see that the corresponding orthogonal polynomials are continuous dual Hahn polynomials.

3. From pfaffians to Hankel determinants

We will obtain sums of squares formulas from the following pfaffian evaluations, the first of which was already given in [R1].

Lemma 3.1. *One has*

$$\operatorname*{pfaff}_{1\leq i,j\leq 2m}\left(\frac{\theta(x_j/x_i)}{\theta(-x_j/x_i)}\right)=\prod_{1\leq i<j\leq 2m}\frac{\theta(x_j/x_i)}{\theta(-x_j/x_i)}, \tag{3.1a}$$

$$\operatorname*{pfaff}_{1\leq i,j\leq 2m+1}\left(1+2\frac{x_j\,\theta'(-x_j/x_i)}{x_i\,\theta(-x_j/x_i)}\right)=\frac{(q)_\infty^{2m}}{(-q)_\infty^{2m}}\prod_{1\leq i<j\leq 2m+1}\frac{\theta(x_j/x_i)}{\theta(-x_j/x_i)}. \tag{3.1b}$$

Remark 3.2. Since $\theta(x;0)=1-x$, Lemma 3.1 reduces to (2.2) when $q=0$. A different elliptic extension of Schur's pfaffian evaluation was recently obtained by Okada [O], see also [R1, Remark 2.1].

We give two proofs of (3.1b). The identity (3.1a) can be proved similarly [R1].

First proof of (3.1b). We start from the Frobenius–Stickelberger determinant [FS]

$$\det_{1\leq i,j\leq n+1}\begin{pmatrix} -\dfrac{x_jy_i\,\theta'(x_jy_i)}{(q)_\infty^2\theta(x_jy_i)} & \begin{matrix}1\\ \vdots\\ 1\end{matrix}\\ -1\quad\cdots\quad -1 & 0\end{pmatrix}$$
$$=\frac{\theta(x_1\cdots x_ny_1\cdots y_n)\prod_{1\leq i<j\leq n}x_jy_j\theta(x_i/x_j)\theta(y_i/y_j)}{\prod_{i,j=1}^n\theta(x_iy_j)}.$$

Choosing $n=2m+1$ and $y_j=-1/x_j$, and moreover adding $1/2(q)_\infty^2$ times the last column to all other columns gives

$$\det_{1\leq i,j\leq 2m+2}\begin{pmatrix} \dfrac{1}{2(q)_\infty^2}\left(1+2\dfrac{x_j\,\theta'(-x_j/x_i)}{x_i\,\theta(-x_j/x_i)}\right) & \begin{matrix}1\\ \vdots\\ 1\end{matrix}\\ -1\qquad\cdots\qquad -1 & 0\end{pmatrix}$$
$$=\frac{1}{4^m(-q)_\infty^{4m}}\prod_{1\leq i<j\leq n}\frac{\theta(x_i/x_i)^2}{\theta(-x_j/x_i)^2}.$$

We now observe that the matrix on the left is skew-symmetric. This is easily proved by differentiating (2.3). Thus, in view of (2.1), we may conclude that (3.1b) holds up to a factor ±1, which is independent of q by continuity. By Remark 3.2, that factor has to be $+1$. $\square$

Second proof of (3.1b). In (2.13b), let $q = e^{-2\pi/h}$ and $x_j = e^{2\pi i z_j}$. Applying the logarithmic derivative of (2.4), that is,

$$2ie^{2\pi ix}\frac{\theta'(e^{2\pi ix};e^{-2\pi/h})}{\theta(e^{2\pi ix};e^{-2\pi/h})} = i + h - 2hx - 2he^{-2\pi hx}\frac{\theta'(e^{-2\pi hx};e^{-2\pi h})}{\theta(e^{-2\pi hx};e^{-2\pi h})},$$

the left-hand side of (2.13b) takes the form

$$\left(\frac{he^{\pi/h}}{2i}\right)^m \operatorname*{pfaff}_{1\leq i,j\leq 2m+1}\left(1 + 2\frac{e^{2\pi hz_j}\,\theta'(-e^{2\pi h(z_j-z_i)};e^{-2\pi h})}{e^{2\pi hz_i}\,\theta(-e^{2\pi h(z_j-z_i)};e^{-2\pi h})} + 2(z_j - z_i)\right).$$

By Lemma 3.3 below, we may subtract $2(z_j - z_i)$ from each matrix element, which gives a pfaffian of the desired type. Similarly applying (2.4) and (2.9) to the right-hand side of (2.13b) completes the proof. □

The following lemma was used above.

Lemma 3.3. *For any odd-dimensional skew-symmetric matrix* (a_{ij})*,*

$$\operatorname*{pfaff}_{1\leq i,j\leq 2m+1}(a_{ij} + b_i - b_j) = \operatorname*{pfaff}_{1\leq i,j\leq 2m+1}(a_{ij}).$$

Proof. Adding multiples of the last row and column to the previous ones gives

$$\det_{1\leq i,j\leq 2m+2}\begin{pmatrix} & & & 1\\ & a_{ij}+b_i-b_j & & \vdots\\ & & & 1\\ -1 & \cdots & -1 & 0\end{pmatrix} = \det_{1\leq i,j\leq 2m+2}\begin{pmatrix} & & & 1\\ & a_{ij} & & \vdots\\ & & & 1\\ -1 & \cdots & -1 & 0\end{pmatrix}.$$

By (2.1), we may extract square roots to conclude that

$$\operatorname*{pfaff}_{1\leq i,j\leq 2m+1}(a_{ij} + b_i - b_j) = \pm \operatorname*{pfaff}_{1\leq i,j\leq 2m+1}(a_{ij}).$$

Since both sides are polynomial expressions, the sign may be determined by letting $b_i \equiv 0$. □

Proceeding in analogy with §2.4, the next step would be to expand the matrix elements in (3.1) as Laurent series in the variables x_i. An important difference from (2.13) is the presence of singularities at $x_i = -x_j$, which precludes Laurent expansion near $x_i = x_j = 1$. To circumvent this difficulty we will use Abel means, for which we recall the notation (2.20).

Lemma 3.4. *If* $|x| = 1$ *and* $x \neq 1$*, then*

$$\frac{(q)_\infty^2\theta(x)}{(-q)_\infty^2\theta(-x)} = \sideset{}{'}\sum_{k\neq 0}\frac{1-q^k}{1+q^k}(-x)^k, \tag{3.2a}$$

$$1 + 2x\frac{\theta'(-x)}{\theta(-x)} = \sideset{}{'}\sum_{k\neq 0}\frac{1+q^k}{1-q^k}(-x)^k. \tag{3.2b}$$

Proof. By (2.6),

$$\frac{(q)_\infty^2\theta(x)}{(-q)_\infty^2\theta(-x)} = 2\sum_{k=-\infty}^{\infty}\frac{(-x)^k}{1+q^k}, \qquad q<|x|<1, \tag{3.3}$$

Replacing x by qx and using (2.3) gives

$$\frac{(q)_\infty^2\theta(x)}{(-q)_\infty^2\theta(-x)} = -2\sum_{k=-\infty}^{\infty}\frac{(-qx)^k}{1+q^k}, \qquad 1<|x|<q^{-1}. \tag{3.4}$$

Letting $f(x)$ denote the left-hand side of (3.2a), we write

$$f(x) = \lim_{t\to 1^-}\frac{f(xt)+f(x/t)}{2}.$$

Expanding $f(xt)$ using (3.3) and $f(x/t)$ using (3.4) yields (3.2a).

As for (3.2b), if $f(x)$ denote its left-hand side then (2.7) gives

$$f(x) = 1 + 2\sum_{k\neq 0}\frac{(-x)^k}{1-q^k}, \qquad q<|x|<1.$$

It is easy to check that $f(1/x) = -f(x)$. Indeed, this is equivalent to the skew-symmetry of the matrix (3.1b). Thus,

$$f(x) = -1 + 2\sum_{k\neq 0}\frac{q^k(-x)^k}{1-q^k}, \qquad 1<|x|<q^{-1}.$$

Similarly as above, we can now deduce (3.2b). □

Applying (3.2a) to the matrix elements of (3.1a) gives

$$\begin{aligned}
&\frac{(q)_\infty^{2m}}{(-q)_\infty^{2m}}\prod_{1\le i<j\le 2m}\frac{\theta(x_j/x_i)}{\theta(-x_j/x_i)} \\
&\qquad = \frac{1}{2^m m!}\sum_{\sigma\in S_{2m}}\operatorname{sgn}(\sigma)\prod_{i=1}^{m}\left(\sideset{}{'}\sum_{k\neq 0}(-1)^k\frac{1-q^k}{1+q^k}\left(\frac{x_{\sigma(2i)}}{x_{\sigma(2i-1)}}\right)^k\right) \\
&\qquad = \frac{1}{2^m m!}\sideset{}{'}\sum_{k_1,\dots,k_m\neq 0}\prod_{i=1}^{m}(-1)^{k_i}\frac{1-q^{k_i}}{1+q^{k_i}}\sum_{\sigma\in S_{2m}}\operatorname{sgn}(\sigma)\prod_{i=1}^{m}\left(\frac{x_{\sigma(2i)}}{x_{\sigma(2i-1)}}\right)^{k_i},
\end{aligned}$$

where we assume that $|x_i| = |x_j|$ and $x_i \neq -x_j$ for all i and j. Multiplying both sides with $\prod_{i<j}(x_i+x_j)/(x_i-x_j)$ and adopting the notation (2.14), this may be

written

$$\frac{(q)_\infty^{2m}}{(-q)_\infty^{2m}}\prod_{1\leq i<j\leq 2m}\frac{(qx_j/x_i,qx_i/x_j)_\infty}{(-qx_j/x_i,-qx_i/x_j)_\infty}=\frac{1}{2^m m!}\prod_{1\leq i<j\leq 2m}(x_i+x_j)$$
$$\times\sum_{k_1,\dots,k_m\neq 0}{}^{'}\prod_{i=1}^m(-1)^{k_i}\frac{1-q^{k_i}}{1+q^{k_i}}S_{(-k_1,k_1,\dots,-k_m,k_m)}(x_1,\dots,x_{2m}).$$

Since the summand is anti-symmetric and even as a function of k_i, and vanishes unless the k_i are all distinct, we may restrict the summation to $k_1>\dots>k_m\geq 1$ if we multiply by $2^m m!$. Moreover,

$$S_{(-k_1,k_1,\dots,-k_m,k_m)}=(-1)^m S_{(k_1,\dots,k_m,-k_m,\dots,-k_1)}.$$

This gives the first half of following result, which should be compared both with the Kac–Wakimoto identities (2.16) and with the closely related expansion given in Theorem 5.5 below. The second half follows similarly from (3.1b).

Theorem 3.5. *If $|x_i|=|x_j|$ and $x_i\neq -x_j$ for all i and j, then*

$$\frac{(q)_\infty^{2m}}{(-q)_\infty^{2m}}\prod_{1\leq i<j\leq 2m}\frac{(qx_j/x_i,qx_i/x_j)_\infty}{(-qx_j/x_i,-qx_i/x_j)_\infty}=\prod_{1\leq i<j\leq 2m}(x_i+x_j)$$
$$\times\lim_{t\to 1^-}\sum_{k_1>\dots>k_m\geq 1}\prod_{i=1}^m t^{k_i}(-1)^{k_i+1}\frac{1-q^{k_i}}{1+q^{k_i}}\,S_{(k_1,\dots,k_m,-k_m,\dots,-k_1)}(x_1,\dots,x_{2m}),\quad(3.5a)$$

$$\frac{(q)_\infty^{2m}}{(-q)_\infty^{2m}}\prod_{1\leq i<j\leq 2m+1}\frac{(qx_j/x_i,qx_i/x_j)_\infty}{(-qx_j/x_i,-qx_i/x_j)_\infty}=\prod_{1\leq i<j\leq 2m+1}(x_i+x_j)$$
$$\times\lim_{t\to 1^-}\sum_{k_1>\dots>k_m\geq 1}\prod_{i=1}^m t^{k_i}(-1)^{k_i}\frac{1+q^{k_i}}{1-q^{k_i}}\,S_{(k_1,\dots,k_m,0,-k_m,\dots,-k_1)}(x_1,\dots,x_{2m+1}).\quad(3.5b)$$

To obtain sums of squares formulas from (3.5), we let $x_i\equiv 1$. Then, the left-hand sides reduce to $\square(-q)^{4m^2}$ and $\square(-q)^{4m(m+1)}$. On the right, (2.17) gives

$$S_{(k_1,\dots,k_m,-k_m,\dots,-k_1)}(1^{2m})=\frac{2^m}{\prod_{i=1}^{2m-1}i!}\prod_{i=1}^m k_i\prod_{1\leq i<j\leq m}(k_j^2-k_i^2)^2,$$

$$S_{(k_1,\dots,k_m,0,-k_m,\dots,-k_1)}(1^{2m+1})=\frac{2^m}{\prod_{i=1}^{2m}i!}\prod_{i=1}^m k_i^3\prod_{1\leq i<j\leq m}(k_j^2-k_i^2)^2.$$

Focusing on (3.5a), we thus obtain

$$\square(q)^{4m^2}=\frac{4^{m^2}}{m!\prod_{i=1}^{2m-1}i!}\sum_{k_1,\dots,k_m=1}^{\infty}{}^{'}\prod_{i=1}^m(-1)^{k_i+1}\frac{1-(-q)^{k_i}}{1+(-q)^{k_i}}\,k_i\prod_{1\leq i<j\leq m}(k_j^2-k_i^2)^2.$$

By Lemma 2.1, this may be written in Hankel determinant form as

$$\square(q)^{4m^2} = \frac{2^{m(2m-1)}}{\prod_{i=1}^{2m-1} i!} \det_{1\le i,j\le m} \left(\nu_0(x^{i+j-2})\right),$$

where

$$\nu_0(f) = 2\sum_{k=1}^{\infty}{}' \frac{1-(-q)^k}{1+(-q)^k} (-1)^{k+1} k f(k^2).$$

Next, writing

$$\frac{1-(-q)^k}{1+(-q)^k} = 1 - 2\frac{(-q)^k}{1+(-q)^k},$$

leads to the decomposition $\nu_0 = \lambda_0 + \mu_0$, where μ_0 is as in (2.22), and

$$\lambda_0(f) = 4\sum_{k=1}^{\infty} \frac{q^k k}{1+(-q)^k} f(-k^2);$$

since the latter sum is convergent, there is no need to take the Abel mean. Using Corollary 2.3 to identify the moments of μ_0, we arrive at the first half of Corollary 3.6.

Similarly, (3.5b) leads to the identity

$$\square(q)^{4m(m+1)} = \frac{2^{m(2m+1)}}{\prod_{i=1}^{2m} i!} \det_{1\le i,j\le m} \left(\nu_1(x^{i+j-2})\right),$$

where

$$\nu_1(f) = 2\sum_{k=1}^{\infty}{}' \frac{1+(-q)^k}{1-(-q)^k} (-1)^k k^3 f(k^2).$$

Writing $\nu_1 = \lambda_1 + \mu_1$, with μ_1 as in (2.22) and

$$\lambda_1(f) = 4\sum_{k=1}^{\infty} \frac{q^k k^3}{1-(-q)^k} f(-k^2).$$

yields the second half of Corollary 3.6. These identities are equivalent to [M3, Theorems 5.3 and 5.5].

Corollary 3.6 (Milne)**.** *One has*

$$\square(q)^{4m^2} = \frac{2^{m(2m-1)}}{\prod_{i=1}^{2m-1} i!} \det_{1\le i,j\le m} \left(t_{i+j-1} + 4(-1)^{i+j} \sum_{k=1}^{\infty} \frac{q^k k^{2i+2j-3}}{1+(-q)^k} \right),$$

$$\square(q)^{4m(m+1)} = \frac{2^{m(2m+1)}}{\prod_{i=1}^{2m} i!} \det_{1\le i,j\le m} \left(t_{i+j} + 4(-1)^{i+j} \sum_{k=1}^{\infty} \frac{q^k k^{2i+2j-1}}{1-(-q)^k} \right),$$

where the numbers t_k are defined in (2.19)*.*

Let $p_k^{(\varepsilon)}(x;q)$ be the monic orthogonal polynomials corresponding to ν_ε. Then, (2.18) gives the following reformulation of Milne's formulas.

Corollary 3.7. *In the notation above,*

$$\Box(q)^{4m^2}=\frac{2^{m(2m-1)}}{\prod_{i=1}^{2m-1}i!}\prod_{i=1}^{m}\|p_{i-1}^{(0)}(x;q)\|^2,$$

$$\Box(q)^{4m(m+1)}=\frac{2^{m(2m+1)}}{\prod_{i=1}^{2m}i!}\prod_{i=1}^{m}\|p_{i-1}^{(1)}(x;q)\|^2.$$

Equivalently,

$$\|p_k^{(0)}(x;q)\|^2=\frac{(2k+1)!(2k)!}{2^{4k+1}}\Box(q)^{8k+4}, \tag{3.6a}$$

$$\|p_k^{(1)}(x;q)\|^2=\frac{(2k+2)!(2k+1)!}{2^{4k+3}}\Box(q)^{8k+8}. \tag{3.6b}$$

It is well-known that any positive definite Hankel determinant can be expressed in terms of orthogonal polynomials. The point here is the explicit expressions for the moment functionals, which might suggest that the polynomials $p_k^{(\varepsilon)}(x;q)$ are of independent interest. In particular, an alternative proof of (3.6) would lead to a new proof of Milne's sums of squares formulas. Note also that $p_k^{(\varepsilon)}(x;q)$ are q-analogues of the polynomials $p_k^{(\varepsilon)}(x;0)$, which, as we shall see in §4.1, are continuous dual Hahn polynomials. However, they are of a different type from the polynomials in the q-Askey Scheme [KS].

4. Correlation functions and sums of squares

To obtain sums of squares formulas from Corollary 3.6, we must expand the determinants as power series in q. In [M3], this is achieved using the Cauchy–Binet formula, which leads to identities involving Schur functions, see Corollary 4.5. We will give a more conceptual derivation of these identities by relating them to correlation functions for continuous dual Hahn polynomials.

4.1. Continuous dual Hahn polynomials.

We need the following alternative expressions for the functionals μ_ε defined in (2.22).

Lemma 4.1. *One has*

$$\mu_\varepsilon(f)=\frac{1}{2}\int_0^\infty\frac{x^{2\varepsilon+1}f(x^2)}{\sinh(\pi x)}\,dx,\qquad \varepsilon=0,\,1.$$

Proof. Write

$$\frac{1}{2}\int_0^\infty\frac{x^{2\varepsilon+1}f(x^2)}{\sinh(\pi x)}\,dx=\lim_{\lambda\to 0^+}\int_{-\infty}^\infty\frac{x^{2\varepsilon+1}e^{i\lambda x}f(x^2)}{\sinh(\pi x)}\,dx$$

and expand as the sum of residues in the upper half-plane. □

Alternatively, Lemma 4.1 can be deduced from Corollary 2.3. One is then reduced to the integral evaluation

$$\int_0^\infty \frac{x^{2k-1}}{\sinh(\pi x)}\,dx = \frac{t_k}{2},$$

which can be found in standard tables, or derived from

$$\int_0^\infty \frac{\sinh(tx)}{\sinh(\pi x)}\,dx = \frac{\tan(t/2)}{2}, \qquad -\pi < t < \pi$$

by expanding both sides as power series in t.

Let $(p_k^{(\varepsilon)})_{k=0}^\infty$ be the monic orthogonal polynomials corresponding to μ_ε, $\varepsilon = 0, 1$. We claim that they may be identified with *continuous dual Hahn polynomials* [KS]. In general, when $a, b, c \geq 0$, one denotes by $(-1)^k S_k(x; a, b, c)$ the monic orthogonal polynomials with respect to

$$f \mapsto \int_0^\infty \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2 f(x^2)\,dx.$$

Using elementary properties of the gamma function, it follows from Lemma 4.1 that

$$p_k^{(0)}(x) = (-1)^k S_k(x; 0, 1/2, 1), \qquad p_k^{(1)}(x) = (-1)^k S_k(x; 1/2, 1, 1).$$

Since $p_k^{(\varepsilon)}(x) = p_k^{(\varepsilon)}(x; 0)$, the case $q = 0$ of Corollary 3.7 reads

$$\prod_{i=1}^m \|p_{i-1}^{(0)}\|^2 = \frac{1}{2^{m(2m-1)}} \prod_{i=1}^{2m-1} i!, \qquad \prod_{i=1}^m \|p_{i-1}^{(1)}\|^2 = \frac{1}{2^{m(2m+1)}} \prod_{i=1}^{2m} i!, \tag{4.1}$$

which agrees with the known expressions for the norms [KS].

We remark that $p_k^{(0)}$ and $p_k^{(1)}$ combine naturally to a single orthogonal system. Indeed, if

$$p_{2k}(x) = p_k^{(0)}(x^2), \qquad p_{2k+1}(x) = x p_k^{(1)}(x^2),$$

then $(p_k(x))_{k=0}^\infty$ are the monic orthogonal polynomials corresponding to

$$\int_{-\infty}^\infty \frac{x f(x)}{\sinh(\pi x)}\,dx.$$

These are Meixner–Pollaczek polynomials; in the notation of [KS]

$$p_k(x) = \frac{k!}{2^k}\, P_k^{(1)}(x; \pi/2).$$

4.2. **Correlation functions.** We recall some general facts about correlation functions, see [R3] for references and further details.

Let

$$\mu(f)=\int f(x)\,d\mu(x)$$

be a positive moment functional and $(p_k(x))_{k=0}^{\infty}$ the corresponding family of monic orthogonal polynomials. Normalizing

$$\Delta(x_1,\dots,x_n)^2\,d\mu(x_1)\cdots d\mu(x_m)$$

to a probability measure, it defines an *orthogonal polynomial ensemble*. Such ensembles arise in several contexts, including the theory of random hermitian matrices [K]. An important role is played by the correlation functions

$$\int \Delta(x_1,\dots,x_n)^2\,d\mu(x_{m+1})\cdots d\mu(x_n),\qquad 0\leq m\leq n.$$

We will normalize these functions as

$$\begin{aligned}C(x)=C_m^n(x_1,\dots,x_m)&=\frac{1}{(n-m)!\prod_{i=1}^n\|p_{i-1}\|^2\Delta(x_1,\dots,x_m)^2}\\&\qquad\times\int\Delta(x_1,\dots,x_n)^2\,d\mu(x_{m+1})\cdots d\mu(x_n).\end{aligned}\tag{4.2}$$

Then, C is a symmetric polynomial.

There are several expressions for correlation functions in terms of orthogonal polynomials, including

$$C(x)=\frac{1}{\|p_{n-1}\|^{2m}\Delta(x)^2}\det_{1\leq i,j\leq m}\left(\frac{p_n(x_i)p_{n-1}(x_j)-p_{n-1}(x_i)p_n(x_j)}{x_i-x_j}\right)\tag{4.3a}$$

$$=\frac{(-1)^{\frac12 m(m-1)}}{\prod_{i=1}^m\|p_{n-i}\|^2\Delta(x)^4}\det_{1\leq i,j\leq 2m}\left(\begin{cases}p_{n-m+j-1}(x_i), & 1\leq i\leq m,\\ p'_{n-m+j-1}(x_i), & m+1\leq i\leq 2m\end{cases}\right)\tag{4.3b}$$

$$=\frac{1}{\Delta(x)^2}\sum_{0\leq k_m<\dots<k_1\leq n-1}\frac{(\det_{1\leq i,j\leq m}(p_{k_i}(x_j)))^2}{\prod_{i=1}^m\|p_{k_i}\|^2},\tag{4.3c}$$

where the diagonal entries in (4.3a) are interpreted as the limit

$$\lim_{x_j\to x_i}\frac{p_n(x_i)p_{n-1}(x_j)-p_{n-1}(x_i)p_n(x_j)}{x_i-x_j}=p_n'(x_i)p_{n-1}(x_i)-p_{n-1}'(x_i)p_n(x_i).$$

To recover Milne's Schur function expansions we need another identity, namely [R3, Proposition 1.8],

$$C(x)=\sum_{\substack{0\leq\lambda_m\leq\dots\leq\lambda_1\leq n-m\\0\leq\mu_m\leq\dots\leq\mu_1\leq n-m}}\frac{(-1)^{\sum_{i=1}^m(\lambda_i+\mu_i)}}{\prod_{i=1}^n\|p_{i-1}\|^2}\det_{i\in[n]\setminus S,j\in[n]\setminus T}(c_{i+j-2})\,s_\lambda(x)s_\mu(x),\tag{4.4}$$

where $c_k = \mu(x^k)$, and where

$$S = \{\lambda_k + m + 1 - k;\ 1 \le k \le m\}, \qquad T = \{\mu_k + m + 1 - k;\ 1 \le k \le m\}.$$

An important relation between Hankel determinants and correlation functions follows from Lemma 2.1 upon replacing μ by $\mu+\lambda$, where we think of μ as positive and λ as arbitrary. Since the integrand is symmetric, we may write

$$\begin{aligned}
&\det_{1\le i,j\le m}\left(\mu(x^{i+j-2}) + \lambda(x^{i+j-2})\right)\\
&\qquad = \frac{1}{m!}\int \Delta(x_1,\dots,x_m)^2\, d(\mu+\lambda)(x_1)\cdots d(\mu+\lambda)(x_m)\\
&\qquad = \sum_{s=0}^{m} \frac{1}{s!(m-s)!}\int \Delta(x_1,\dots,x_m)^2\, d\lambda(x_1)\cdots d\lambda(x_s) d\mu(x_{s+1})\cdots d\mu(x_m),
\end{aligned}$$

where the integral over the last $m-s$ variables is a correlation function. We thus arrive at the following result, which is a standard tool of random matrix theory; see e.g. [J, §2].

Lemma 4.2. *In the notation above,*

$$\begin{aligned}
&\det_{1\le i,j\le m}\left(\mu(x^{i+j-2}) + \lambda(x^{i+j-2})\right)\\
&\qquad = \prod_{i=1}^{m} \|p_{i-1}\|^2 \sum_{s=0}^{m} \frac{1}{s!}\int \Delta(x_1,\dots,x_s)^2\, C_s^m(x_1,\dots,x_s)\, d\lambda(x_1)\cdots d\lambda(x_s).
\end{aligned}$$

4.3. **Milne's sums of squares formulas.** By Lemma 4.2, the Hankel determinants of Corollary 3.6 may be expressed in terms of correlation functions for continuous dual Hahn polynomials. If we let $C_m^{n,\varepsilon}$ denote the correlation function defined by choosing $\mu = \mu_\varepsilon$ in (4.2), then applying Lemma 4.2 with $\mu = \mu_0$ and $\lambda = \lambda_0$ yields, using also (4.1),

$$\square(q)^{4m^2} = \sum_{s=0}^{m} \frac{4^s}{s!} \sum_{k_1,\dots,k_s=1}^{\infty} \prod_{i=1}^{s} \frac{q^{k_i} k_i}{1+(-q)^{k_i}} \prod_{1\le i<j\le s} (k_i^2 - k_j^2)^2\, C_s^{m,0}(-k_1^2,\dots,-k_s^2).$$

It is understood that the sum over $k_1,\dots,k_s$ equals 1 if $s=0$. Similarly as in Theorem 3.5, we may restrict the summation to $k_1 > \cdots > k_s$ if we multiply by $s!$, thus obtaining the first half of Corollary 4.3. The second half follows similarly, choosing $\mu = \mu_1$ and $\lambda = \lambda_1$ in Lemma 4.2.

Corollary 4.3. *One has*

$$\square(q)^{4m^2} = \sum_{s=0}^{m} 4^s \sum_{k_1>\cdots>k_s\ge 1} \prod_{i=1}^{s} \frac{q^{k_i} k_i}{1+(-q)^{k_i}} \prod_{1\le i<j\le s} (k_i^2 - k_j^2)^2\, C_s^{m,0}(-k_1^2,\dots,-k_s^2),$$

$$\square(q)^{4m(m+1)}$$
$$= \sum_{s=0}^{m} 4^s \sum_{k_1>\cdots>k_s\geq 1} \prod_{i=1}^{s} \frac{q^{k_i}k_i^3}{1-(-q)^{k_i}} \prod_{1\leq i<j\leq s} (k_i^2-k_j^2)^2\, C_s^{m,1}(-k_1^2,\dots,-k_s^2).$$

Expanding

$$\frac{q^k}{1+(-q)^k} = \sum_{l=1}^{\infty}(-1)^{(k-1)(l-1)}q^{kl}, \qquad \frac{q^k}{1-(-q)^k} = \sum_{l=1}^{\infty}(-1)^{k(l-1)}q^{kl},$$

gives the following sums of squares formulas. We have excluded the terms with $s=0$, since they only contribute to the trivial coefficients corresponding to $n=0$.

Theorem 4.4. *For $n>0$,*

$$\square_{4m^2}(n) = \sum_{s=1}^{m} 4^s \sum_{\substack{k_1l_1+\cdots+k_sl_s=n\\ k_1>\cdots>k_s\geq 1\\ l_1,\dots,l_s\geq 1}} \prod_{i=1}^{s}(-1)^{(k_i-1)(l_i-1)}k_i \prod_{1\leq i<j\leq s}(k_i^2-k_j^2)^2$$
$$\times\, C_s^{m,0}(-k_1^2,\dots,-k_s^2),$$

$$\square_{4m(m+1)}(n) = \sum_{s=1}^{m} 4^s \sum_{\substack{k_1l_1+\cdots+k_sl_s=n\\ k_1>\cdots>k_s\geq 1\\ l_1,\dots,l_s\geq 1}} \prod_{i=1}^{s}(-1)^{k_i(l_i-1)}k_i^3 \prod_{1\leq i<j\leq s}(k_i^2-k_j^2)^2$$
$$\times\, C_s^{m,1}(-k_1^2,\dots,-k_s^2).$$

Expressing the correlation functions as in (4.3) gives explicit versions of these sums of squares formulas in terms of continuous dual Hahn polynomials. Using instead (4.4), together with the expressions (2.23) for the moments and (4.1) for the norms, yields the following identities, which are equivalent to [M3, Theorems 7.1 and 7.2].

Corollary 4.5 (Milne)**.** *For $n>0$,*

$$\square_{4m^2}(n) = \frac{2^{m(2m-1)}}{\prod_{i=1}^{2m-1} i!} \sum_{s=1}^{m} 4^s \sum_{\substack{k_1l_1+\cdots+k_sl_s=n\\ k_1>\cdots>k_s\geq 1\\ l_1,\dots,l_s\geq 1}} \prod_{i=1}^{s}(-1)^{(k_i-1)(l_i-1)}k_i \prod_{1\leq i<j\leq s}(k_i^2-k_j^2)^2$$
$$\times \sum_{\substack{0\leq\lambda_s\leq\cdots\leq\lambda_1\leq m-s\\ 0\leq\mu_s\leq\cdots\leq\mu_1\leq m-s}} \det_{i\in[m]\setminus S,\, j\in[m]\setminus T}(t_{i+j-1})\, s_\lambda(k_1^2,\dots,k_s^2)s_\mu(k_1^2,\dots,k_s^2),$$

$$\square_{4m(m+1)}(n) = \frac{2^{m(2m+1)}}{\prod_{i=1}^{2m} i!} \sum_{s=1}^{m} 4^s \sum_{\substack{k_1l_1+\dots+k_sl_s=n\\ k_1>\dots>k_s\geq 1\\ l_1,\dots,l_s\geq 1}} \prod_{i=1}^{s} (-1)^{k_i(l_i-1)} k_i^3 \prod_{1\leq i<j\leq s} (k_i^2-k_j^2)^2$$
$$\times \sum_{\substack{0\leq\lambda_s\leq\dots\leq\lambda_1\leq m-s\\ 0\leq\mu_s\leq\dots\leq\mu_1\leq m-s}} \det_{i\in[m]\setminus S,\, j\in[m]\setminus T}(t_{i+j})\, s_\lambda(k_1^2,\dots,k_s^2) s_\mu(k_1^2,\dots,k_s^2),$$

where t_k is as in (2.19) *and*

$$S = \{\lambda_k + s + 1 - k;\ 1\leq k\leq s\}, \qquad T = \{\mu_k + s + 1 - k;\ 1\leq k\leq s\}.$$

5. Relation to Schur Q-polynomials

In [R2], we showed that the correlation functions appearing in Theorem 4.4 also arise from generalized Schur Q-polynomials by specializing all variables to 1. In this section, we give a slightly different proof of Milne's formulas, where Schur Q-polynomials appear naturally.

5.1. **Schur Q-polynomials.** When $m\leq n$ and $\lambda\in\mathbb{Z}^m$, we write

$$Q_\lambda(x_1,\dots,x_n) = 2^m \sum_{\sigma\in S_n/S_{n-m}} \sigma\left(x_1^{\lambda_1}\cdots x_m^{\lambda_m} \prod_{\substack{1\leq i\leq m\\ 1\leq i<j\leq n}} \frac{x_i+x_j}{x_i-x_j}\right). \tag{5.1}$$

Here, S_n acts by permuting the variables $x_1,\dots,x_n$ and S_{n-m} is the subgroup acting on $x_{m+1},\dots,x_n$. When $\lambda_1\geq\lambda_2\geq\dots\geq\lambda_m>0$, this is a standard definition of Schur Q-polynomials [Ma]. Note that there is no analogue of (2.15), so the case when some λ_i are negative cannot immediately be reduced to the classical situation.

We will not work with (5.1) directly, but rather with the alternative expression [R2, Lemma A.1]

$$Q_\lambda(x_1,\dots,x_n)$$
$$= \frac{2^{m-k}}{k!} \prod_{1\leq i<j\leq n} \frac{x_i+x_j}{x_i-x_j} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{m} x_{\sigma(i)}^{\lambda_i} \prod_{i=1}^{k} \frac{x_{\sigma(m+2i-1)}-x_{\sigma(m+2i)}}{x_{\sigma(m+2i-1)}+x_{\sigma(m+2i)}}, \tag{5.2}$$

where k is the integral part of $(n-m)/2$. In analogy with (2.14), the right-hand side can be rewritten as a quotient of two pfaffians [N, Eq. (A12)]; however, the form given here is more convenient for our purposes.

One of the main results of [R2] is the identification of $Q_\lambda(1,q,\dots,q^{n-1})$ with generalized Christoffel–Darboux kernels for continuous q-Jacobi polynomials. In the case when $q=1$ and $\lambda=(\lambda_1,\dots,\lambda_m,-\lambda_m,\dots,-\lambda_1)$, the continuous q-Jacobi polynomials reduce to continuous dual Hahn polynomials and the generalized Christoffel–Darboux kernels to correlation functions. To be precise, noting that

the kernels $K_m^{n,\varepsilon}$ featuring in [R2] are related to the kernels $C_m^{n,\varepsilon}$ introduced in §4.3 through

$$C_m^{n,\varepsilon}(x_1,\dots,x_m)=K_m^{n,\varepsilon}(x_1,\dots,x_m,x_1,\dots,x_m),$$

it follows from [R2, Corollary 5.11] that

$$\begin{aligned}Q_{(\lambda_1,\dots,\lambda_m,-\lambda_m,\dots,-\lambda_1)}&(1^{2n})\\&=8^m\prod_{i=1}^m\lambda_i\prod_{1\leq i<j\leq m}(\lambda_i^2-\lambda_j^2)^2\,C_m^{n,0}(-\lambda_1^2,\dots,-\lambda_m^2),\end{aligned}\tag{5.3a}$$

$$\begin{aligned}Q_{(\lambda_1,\dots,\lambda_m,-\lambda_m,\dots,-\lambda_1)}&(1^{2n+1})\\&=(-1)^m8^m\prod_{i=1}^m\lambda_i^3\prod_{1\leq i<j\leq m}(\lambda_i^2-\lambda_j^2)^2\,C_m^{n,1}(-\lambda_1^2,\dots,-\lambda_m^2).\end{aligned}\tag{5.3b}$$

5.2. **A second proof of Milne's formulas.** We return to the pfaffian evaluations (3.1). In §3, we handled the singularities at $x_i=-x_j$ by introducing Abel means. We will now use a slightly different approach, removing the singularity by subtracting a rational function. This corresponds precisely to the previous decomposition $\mu_\varepsilon=\lambda_\varepsilon+\nu_\varepsilon$ of functionals, so the two ideas are in fact intimately related.

Lemma 5.1. *For $q<|x|<q^{-1}$,*

$$\frac{(q)_\infty^2\theta(x)}{(-q)_\infty^2\theta(-x)}=\frac{1-x}{1+x}+2\sum_{k=1}^\infty\frac{(-q)^k(x^{-k}-x^k)}{1+q^k},\tag{5.4a}$$

$$1+2x\frac{\theta'(-x)}{\theta(-x)}=\frac{1-x}{1+x}+2\sum_{k=1}^\infty\frac{(-q)^k(x^k-x^{-k})}{1-q^k}.\tag{5.4b}$$

Proof. Subtracting (2.21) from (3.3) gives (5.4a) for $q<|x|<1$. By analytic continuation, it holds also for $1\leq|x|<q^{-1}$. The equation (5.4b) follows similarly from (2.7). ☐

Applying (5.4) to the matrix elements in (3.1) leads to pfaffians of the form $\operatorname{pfaff}(A-A^t+B)$. The following lemma will be useful.

Lemma 5.2. *One has*

$$\begin{aligned}\operatorname*{pfaff}_{1\leq i,j\leq m}&(a_{ij}-a_{ji}+b_{ij})\\&=\frac{1}{M!}\sum_{s=0}^M\frac{1}{2^{M-s}}\binom{M}{s}\sum_{\sigma\in S_m}\operatorname{sgn}(\sigma)\prod_{i=1}^s a_{\sigma(2i-1),\sigma(2i)}\prod_{i=s+1}^M b_{\sigma(2i-1),\sigma(2i)},\end{aligned}$$

where M is the integral part of $m/2$ and $(b_{ij})_{i,j=1}^m$ is skew-symmetric.

Proof. Consider first

$$\begin{aligned}\operatorname*{pfaff}_{1\leq i,j\leq m}(a_{ij}+b_{ij}) &= \frac{1}{2^M M!}\sum_{\tau\in S_m}\operatorname{sgn}(\tau)\prod_{i=1}^{M}(a_{\tau(2i-1),\tau(2i)}+b_{\tau(2i-1),\tau(2i)})\\ &= \frac{1}{2^M M!}\sum_{S\subseteq[M]}\sum_{\tau\in S_m}\operatorname{sgn}(\tau)\prod_{i\in S}a_{\tau(2i-1),\tau(2i)}\prod_{i\in[M]\setminus S}b_{\tau(2i-1),\tau(2i)},\end{aligned}$$

where $[M]=\{1,2,\dots,M\}$. Choose $\sigma\in S_m$ so that

$$\prod_{i\in S}a_{\tau(2i-1),\tau(2i)}\prod_{i\in[M]\setminus S}b_{\tau(2i-1),\tau(2i)}=\prod_{i=1}^{s}a_{\sigma(2i-1),\sigma(2i)}\prod_{i=s+1}^{M}b_{\sigma(2i-1),\sigma(2i)},$$

where $s=|S|$. This can be done so that, for fixed S, $\tau\mapsto\sigma$ is a bijection. Moreover, $\operatorname{sgn}(\sigma)=\operatorname{sgn}(\tau)$. Rewriting the sum in terms of σ and s yields

$$\operatorname*{pfaff}_{1\leq i,j\leq m}(a_{ij}+b_{ij})=\frac{1}{2^M M!}\sum_{s=0}^{M}\binom{M}{s}\sum_{\sigma\in S_m}\operatorname{sgn}(\sigma)\prod_{i=1}^{s}a_{\sigma(2i-1),\sigma(2i)}\prod_{i=s+1}^{M}b_{\sigma(2i-1),\sigma(2i)}.$$

Replacing a_{ij} by $a_{ij}-a_{ji}$, the sign changes obtained when expanding the product $\prod_{i=1}^{s}(a_{\sigma(2i-1),\sigma(2i)}-a_{\sigma(2i),\sigma(2i-1)})$ are compensated by the factor $\operatorname{sgn}(\sigma)$, thus yielding the desired result. $\square$

If $a_{ij}=\sum_{k=1}^{\infty}c_k(x_i/x_j)^k$ and $b_{ij}=(x_i-x_j)/(x_i+x_j)$, Lemma 5.2 gives

$$\begin{aligned}&\operatorname*{pfaff}_{1\leq i,j\leq m}\left(\frac{x_i-x_j}{x_i+x_j}+\sum_{k=1}^{\infty}c_k\left((x_i/x_j)^k-(x_j/x_i)^k\right)\right)\\ &=\frac{1}{M!}\sum_{s=0}^{M}\frac{1}{2^{M-s}}\binom{M}{s}\sum_{k_1,\dots,k_s=1}^{\infty}\sum_{\sigma\in S_m}\prod_{i=1}^{s}c_{k_i}x_{\sigma(2i-1)}^{k_i}x_{\sigma(2i)}^{-k_i}\prod_{i=s+1}^{M}\frac{x_{\sigma(2i-1)}-x_{\sigma(2i)}}{x_{\sigma(2i-1)}+x_{\sigma(2i)}},\end{aligned}$$

where, as before, the sum over $k_1,\dots,k_s$ equals 1 when $s=0$. By (5.2), this may be written

$$\prod_{1\leq i<j\leq m}\frac{x_i-x_j}{x_i+x_j}\sum_{s=0}^{M}\frac{1}{4^s s!}\sum_{k_1,\dots,k_s=1}^{\infty}\prod_{i=1}^{s}c_{k_i}\,Q_{(k_1,-k_1,\dots,k_s,-k_s)}(x_1,\dots,x_m).$$

Similarly as for Theorem 3.5, we exploit the symmetry in k_i to write the result as follows.

Proposition 5.3. *One has*

$$\operatorname*{pfaff}_{1\leq i,j\leq m}\left(\frac{x_i-x_j}{x_i+x_j}+\sum_{k=1}^{\infty}c_k\left((x_i/x_j)^k-(x_j/x_i)^k\right)\right)$$
$$=\prod_{1\leq i<j\leq m}\frac{x_i-x_j}{x_i+x_j}\sum_{s=0}^{M}\frac{1}{4^s}\sum_{k_1>k_2>\dots>k_s\geq 1}\prod_{i=1}^{s}c_{k_i}\,Q_{(k_1,\dots,k_s,-k_s,\dots,-k_1)}(x_1,\dots,x_m),$$

where $(c_k)_{k=1}^{\infty}$ is a scalar sequence such that the series converge absolutely and M is the integral part of $m/2$.

Remark 5.4. More generally, the same proof gives

$$\operatorname*{pfaff}_{1\leq i,j\leq m}\left(\frac{x_i-x_j}{x_i+x_j}+\sum_{k,l=-\infty}^{\infty}c_{kl}\left(x_i^kx_j^l-x_j^kx_i^l\right)\right)=\prod_{1\leq i<j\leq m}\frac{x_i-x_j}{x_i+x_j}$$
$$\times\sum_{s=0}^{M}\frac{(-1)^{\frac12 s(s-1)}}{4^s s!}\sum_{k_1,\dots,k_s,l_1,\dots,l_s=-\infty}^{\infty}\prod_{i=1}^{s}c_{k_il_i}\,Q_{(k_1,\dots,k_s,l_1,\dots,l_s)}(x_1,\dots,x_m).$$

We apply Proposition 5.3 when m is even and $c_k=2(-q)^k/(1+q^k)$, and when m is odd and $c_k=-2(-q)^k/(1-q^k)$. Then, by Lemma 5.1, the pfaffians are of the form (3.1). We thus arrive at the following result.

Theorem 5.5. *If $q<|x_j/x_i|<q^{-1}$ for all i and j, then*

$$\frac{(q)_\infty^{2m}}{(-q)_\infty^{2m}}\prod_{1\leq i<j\leq 2m}\frac{(qx_j/x_i,qx_i/x_j)_\infty}{(-qx_j/x_i,-qx_i/x_j)_\infty}$$
$$=\sum_{s=0}^{m}\frac{1}{2^s}\sum_{k_1>\dots>k_s\geq 1}\prod_{i=1}^{s}\frac{(-q)^{k_i}}{1+q^{k_i}}\,Q_{(k_1,\dots,k_s,-k_s,\dots,-k_1)}(x_1,\dots,x_{2m}),$$

$$\frac{(q)_\infty^{2m}}{(-q)_\infty^{2m}}\prod_{1\leq i<j\leq 2m+1}\frac{(qx_j/x_i,qx_i/x_j)_\infty}{(-qx_j/x_i,-qx_i/x_j)_\infty}$$
$$=\sum_{s=0}^{m}\frac{(-1)^s}{2^s}\sum_{k_1>\dots>k_s\geq 1}\prod_{i=1}^{s}\frac{(-q)^{k_i}}{1-q^{k_i}}\,Q_{(k_1,\dots,k_s,-k_s,\dots,-k_1)}(x_1,\dots,x_{2m+1}).$$

Remark 5.6. It would be interesting to find an algebraic interpretation of Theorem 5.5, beyond the link to denominator formulas for queer affine superalgebras via modular duality. Plausibly, the work of Sergeev [S1, S2], where Schur Q-polynomials arise as characters and spherical functions for queer superalgebras, is relevant for finding such a relation.

Specializing as before $x_i\equiv 1$ and replacing q by $-q$, Theorem 5.5 reduces to the following identities.

Corollary 5.7. *One has*

$$\square(q)^{4m^2} = \sum_{s=0}^{m} \frac{1}{2^s} \sum_{k_1>\dots>k_s\geq 1} \prod_{i=1}^{s} \frac{q^{k_i}}{1+(-q)^{k_i}}\, Q_{(k_1,\dots,k_s,-k_s,\dots,-k_1)}(1^{2m}),$$

$$\square(q)^{4m(m+1)} = \sum_{s=0}^{m} \frac{(-1)^s}{2^s} \sum_{k_1>\dots>k_s\geq 1} \prod_{i=1}^{s} \frac{q^{k_i}}{1-(-q)^{k_i}}\, Q_{(k_1,\dots,k_s,-k_s,\dots,-k_1)}(1^{2m+1}).$$

By (5.3), this is equivalent to Corollary 4.3, and thus also to Milne's sums of squares formulas.

6. A new formula for $2m^2$ squares

As we have seen, Milne's formulas for $4m^2$ and $4m(m+1)$ squares arise from the pfaffian evaluations (3.1) as all variables $x_i \to 1$. In view of the results for triangular numbers in [R1], one would expect that more general formulas for $4m^2/d$ squares, when $d \mid 2m$, and $4m(m+1)/d$ squares, when $d \mid 2m$ or $d \mid 2(m+1)$, may be obtained by letting x_i tend to suitable fractional powers of q. We have made some preliminary investigations in this direction, but, unless one can simplify the arguments, the results seem very complicated. For the benefit of the interested reader, we state without proof a particularly simple special case, a $2m^2$ squares formula which we consider to be the natural analogue of (1.5). It can be obtained from (3.1a) in the limit when $x_1,\dots,x_m \to 1$, $x_{m+1},\dots,x_{2m} \to \sqrt{q}$.

To state the result we need to introduce the Schur-type polynomials

$$\mathbb{P}^{(\varepsilon)}_{n^m}(x_1,\dots,x_m) = \frac{\det_{1\leq i,j\leq m}(p^{(\varepsilon)}_{n+j-1}(x_i))}{\Delta(x)},$$

where $p^{(\varepsilon)}_k$ are as in §4.1. They generalize the correlation functions of §4.3, since

$$C^{n,\varepsilon}_m(x_1,\dots,x_m) = \frac{1}{\prod_{i=1}^m \|p^{(\varepsilon)}_{n-i}\|^2}\, \mathbb{P}^{(\varepsilon)}_{(n-m)^{2m}}(x_1,\dots,x_m,x_1,\dots,x_m).$$

Moreover, if $n-m = 2k+\varepsilon$, with k a non-negative integer and $\varepsilon \in \{0,1\}$, we have the following generalization of (5.3) [R2, Corollary 5.11]:

$$Q_{(\lambda_1,\dots,\lambda_m)}(1^n) = \frac{2^{\frac12 m(2n+1-m)}(-1)^{km}}{\prod_{i=1}^m (n-i)!} \prod_{i=1}^{m} \lambda_i^{\varepsilon} \prod_{1\leq i<j\leq m} (\lambda_i-\lambda_j)\, \mathbb{P}^{(\varepsilon)}_{k^m}(-\lambda_1^2,\dots,-\lambda_m^2).$$

Theorem 6.1. *One has*

$$
\begin{aligned}
\square_{2m^2}(n) = & \sum_{\substack{s_0,s_1,s_2\geq 0\\ s_0+2s_1\leq m,\, s_2\leq s_1}} (-1)^{(m+1)(s_1+s_2)}(2-\delta_{s_1s_2})\\
& \times \frac{2^{(s_0+s_1+s_2)(2m+1)-\frac12((s_0+2s_1)^2+(s_0+2s_2)^2)}}{\prod_{i=1}^{s_0+2s_1}(m-i)!\prod_{i=1}^{s_0+2s_2}(m-i)!}\\
& \times \sum_{\substack{k_1>\dots>k_{t_0}\geq 1,\, k_{t_0+1}>\dots>k_{s_0}\geq 1,\, l_1,\dots,l_{s_0} \text{ odd positive}\\ k_1'>\dots>k_{s_1}'\geq 1,\, l_1',\dots,l_{s_1}' \text{ even positive}\\ k_1''>\dots>k_{s_2}''\geq 1,\, l_1'',\dots,l_{s_2}'' \text{ even positive}\\ k_1l_1+\dots+k_{s_0}l_{s_0}+k_1'l_1'+\dots+k_{s_1}'l_{s_1}'+k_1''l_1''+\dots+k_{s_2}''l_{s_2}''=n}} \prod_{i=1}^{s_0}(-1)^{\frac12(l_i-1)}\prod_{i=1}^{t_0}k_i^{2+2\varepsilon}\prod_{i=t_0+1}^{s_0}k_i^{2\varepsilon}\\
& \times \prod_{i=1}^{s_1}(-1)^{k_i'+\frac12 l_i'}(k_i')^{1+2\varepsilon}\prod_{i=1}^{s_2}(-1)^{k_i''+\frac12 l_i''}(k_i'')^{1+2\varepsilon}\\
& \times \prod_{\substack{1\leq i<j\leq t_0\\ \text{or } t_0+1\leq i<j\leq s_0}}(k_j^2-k_i^2)^2\prod_{1\leq i\leq s_0,\,1\leq j\leq s_1}(k_i^2-(k_j')^2)\prod_{1\leq i\leq s_0,\,1\leq j\leq s_2}(k_i^2-(k_j'')^2)\\
& \times \prod_{1\leq i<j\leq s_1}((k_j')^2-(k_i')^2)^2\prod_{1\leq i<j\leq s_2}((k_j'')^2-(k_i'')^2)^2\\
& \times \mathbb{P}^{(\varepsilon)}_{t_1^{s_0+2s_1}}(-k_1^2,\dots,-k_{s_0}^2,-(k_1')^2,-(k_1')^2,\dots,-(k_{s_1}')^2,-(k_{s_1}')^2)\\
& \times \mathbb{P}^{(\varepsilon)}_{t_2^{s_0+2s_2}}(-k_1^2,\dots,-k_{s_0}^2,-(k_1'')^2,-(k_1'')^2,\dots,-(k_{s_2}'')^2,-(k_{s_2}'')^2),
\end{aligned}
$$

where $\varepsilon=0$ if $m-s_0$ is even and 1 else, and where t_0, t_1 and t_2 denote the integral part of $s_0/2$, $(m-s_0-2s_1)/2$ and $(m-s_0-2s_2)/2$, respectively.

As examples, we work out the cases $m=1$, 2, 3. We observe that the term $s=(s_0,s_1,s_2)=(0,0,0)$ only contributes to the trivial case when $n=0$, so we assume that $n>0$. Note also that when $m=1$ or 2, all Schur-type polynomials $\mathbb{P}_{k^l}$ that appear have either $k=0$ or $l=0$, and thus equal 1.

2 squares: When $m=1$, the outer sum has only one non-trivial term, $s=(1,0,0)$. We recover the two squares formula (1.1a).

8 squares: When $m=2$, we obtain an eight squares formula, which is more complicated than (1.1c) but still interesting to discuss. We have four non-trivial terms, $s=(0,1,0)$, $(1,0,0)$, $(0,1,1)$ and $(2,0,0)$, giving

$$
\begin{aligned}
\square_8(n) = {} & 16\sum_{k_1'l_1'=n,\, l_1' \text{ even}}(-1)^{k_1'-1+\frac12 l_1'}k_1' + 16\sum_{k_1l_1=n,\, l_1 \text{ odd}}(-1)^{\frac12(l_1-1)}k_1^2\\
& + 64\sum_{\substack{k_1'l_1'+k_1''l_1''=n\\ l_1' \text{ and } l_1'' \text{ even}}}(-1)^{k_1'+\frac12 l_1'+k_1''+\frac12 l_1''}k_1'k_1'' + 64\sum_{\substack{k_1l_1+k_2l_2=n\\ l_1 \text{ and } l_2 \text{ odd}}}(-1)^{\frac12(l_1+l_2-2)}k_1^2.
\end{aligned}
$$

Equivalently,

$$\square(q)^8 = 1 + 16\sum_{k=1}^{\infty}\frac{(-1)^k k q^{2k}}{1+q^{2k}} + 16\sum_{k=1}^{\infty}\frac{k^2q^k}{1+q^{2k}} \\ + 64\left(\sum_{k=1}^{\infty}\frac{(-1)^k k q^{2k}}{1+q^{2k}}\right)^2 + 64\sum_{k=1}^{\infty}\frac{k^2q^k}{1+q^{2k}}\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}}.$$

Computing the Lambert series using (2.10a), (2.10b) and (2.11) gives after simplification

$$\square(q)^8 = \square(-q^2)^8 + 16q\triangle(q^2)^4\square(q)^4.$$

Applying $\square(-q^2)^2 = \square(q)\square(-q)$, which is easily verified either from (2.8) or from the definition, we recover Jacobi's quartic identity (2.12).

18 squares: When $m = 2$, we have seven non-trivial terms, $s = (0,1,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, $(2,0,0)$, $(1,1,1)$ and $(3,0,0)$. The second and fourth term involve the polynomial $\mathbb{P}_{1^1}^{(0)}(x) = p_1^{(0)}(x) = x - \frac{1}{2}$, while all other Schur-type factors are trivial. We thus obtain

$$\begin{aligned}
\square_{18}(n) = 2^5 &\sum_{k_1'l_1'=n,\, l_1' \text{ even}} (-1)^{k_1'+\frac{1}{2}l_1'}(k_1')^3 \\
&+ 2^4 \sum_{k_1l_1=n,\, l_1 \text{ odd}} (-1)^{\frac{1}{2}(l_1-1)}\left(k_1^2+\frac{1}{2}\right)^2 \\
&+ 2^8 \sum_{\substack{k_1'l_1'+k_1''l_1''=n \\ l_1' \text{ and } l_1'' \text{ even}}} (-1)^{k_1'+\frac{1}{2}l_1'+k_1''+\frac{1}{2}l_1''}(k_1')^3(k_1'')^3 \\
&+ 2^8 \sum_{\substack{k_1l_1+k_1'l_1'=n \\ l_1 \text{ odd and } l_1' \text{ even}}} (-1)^{\frac{1}{2}(l_1-1)+k_1'-1+\frac{1}{2}l_1'}\left(k_1^2+\frac{1}{2}\right)k_1' \\
&+ 2^8 \sum_{\substack{k_1l_1+k_2l_2=n \\ l_1 \text{ and } l_2 \text{ odd}}} (-1)^{\frac{1}{2}(l_1+l_2-2)}k_1^4k_2^2 \\
&+ 2^{10} \sum_{\substack{k_1l_1+k_1'l_1'+k_1''l_1''=n \\ l_1 \text{ odd},\, l_1',l_1'' \text{ even}}} (-1)^{\frac{1}{2}(l_1-1)+k_1'+\frac{1}{2}l_1'+k_1''+\frac{1}{2}l_1''}k_1'k_1'' \\
&+ 2^{10} \sum_{\substack{k_1l_1+k_2l_2+k_3l_3=n \\ l_1,l_2,l_3 \text{ odd}}} (-1)^{\frac{1}{2}(l_1+l_2+l_3-3)}k_1^2(k_2^2-k_3^2)^2.
\end{aligned}$$

Appendix. Ono's sums of squares formulas

In this Appendix we show that Ono's sums of squares formulas [On] are equivalent to those of Milne. Using the notation of Ono, we define $A_m^{\pm}$ by

$$\prod_{i=1}^{m} x_i \prod_{1\leq i<j\leq m}(x_j^2-x_i^2)^2=\sum_{\lambda=(a_1,\dots,a_m)} A_m^+(\lambda)\, x_1^{a_1}\cdots x_m^{a_m},$$

$$\prod_{i=1}^{m} x_i^3 \prod_{1\leq i<j\leq m}(x_j^2-x_i^2)^2=\sum_{\lambda=(a_1,\dots,a_m)} A_m^-(\lambda)\, x_1^{a_1}\cdots x_m^{a_m}.$$

Moreover, we need the modular forms

$$E^+(2k)=2^{4k-1}\left(\frac{(-1)^k|B_{2k}|}{4k}+\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^{4n}\right) - 2^{2k-1}\left(\frac{(-1)^k|B_{2k}|}{4k}+\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^{n}\right),$$

$$E^-(2k)=2^{2k}\left(\frac{(-1)^k|B_{2k}|}{4k}+\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^{2n}\right) - \left(\frac{(-1)^k|B_{2k}|}{4k}+\sum_{n=1}^{\infty}(-1)^n\sigma_{2k-1}(n)q^{n}\right),$$

where $\sigma_k(n)=\sum_{d|n}d^k$. Compared to [On], we have replaced q by $-q$, and used that $B_{2i}=(-1)^{i-1}|B_{2i}|$. In generating function form, Ono's result then takes the following form [On, Theorem 1].

Theorem A.2 (Ono)**.** *In the notation above,*

$$\Box(q)^{4m^2}=\frac{(-1)^m4^m}{m!\prod_{i=1}^{2m-1}i!}\sum_{\lambda=(a_1,\dots,a_m)}A_m^+(\lambda)E^+(a_1+1)\cdots E^+(a_m+1),$$

$$\Box(q)^{4m(m+1)}=\frac{2^{2m^2+3m}}{m!\prod_{i=1}^{2m}i!}\sum_{\lambda=(a_1,\dots,a_m)}A_m^-(\lambda)E^-(a_1+1)\cdots E^-(a_m+1).$$

We claim that Theorem A.2 is equivalent to Corollary 3.6. To see this we observe that, in the notation of Lemma 2.1,

$$A_m^+(a_1,\dots,a_m)=C(k_1,\dots,k_m),\qquad a_i=2k_i+1,$$

$$A_m^-(a_1,\dots,a_m)=C(k_1,\dots,k_m),\qquad a_i=2k_i+3.$$

Thus, Theorem A.2 can be written in Hankel determinant form as

$$\Box(q)^{4m^2}=\frac{(-1)^m4^m}{\prod_{i=1}^{2m-1}i!}\det_{1\leq i,j\leq m}(E^+(2i+2j-2)),$$

$$\Box(q)^{4m(m+1)} = \frac{2^{2m^2+3m}}{\prod_{i=1}^{2m} i!} \det_{1\le i,j\le m} (E^-(2i+2j)).$$

Our claim would now would follow from the identities

$$E^+(2k) = (-1)^k 2^{2k-3} \left(\frac{(4^k-1)|B_{2k}|}{k} + 4(-1)^{k+1} \sum_{n=1}^{\infty} \frac{q^n n^{2k-1}}{1+(-q)^n} \right),$$

$$E^+(2k) = \frac{(-1)^k}{4} \left(\frac{(4^k-1)|B_{2k}|}{k} + 4(-1)^{k} \sum_{n=1}^{\infty} \frac{q^n n^{2k-1}}{1-(-q)^n} \right).$$

Cancelling the terms involving Bernoulli numbers, we are reduced to proving that

$$\sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n - 4^k \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{4n} = \sum_{n=1}^{\infty} \frac{q^n n^{2k-1}}{1+(-q)^n},$$

$$4^k \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{2n} + \sum_{n=1}^{\infty} (-1)^{n-1} \sigma_{2k-1}(n) q^n = \sum_{n=1}^{\infty} \frac{q^n n^{2k-1}}{1-(-q)^n},$$

or, equivalently, replacing q by $-q$ in the second identity, to the elementary identities

$$\sum_{l,m\ge1} l^{2k-1} q^{lm} - 2 \sum_{l,m\ge1} (2l)^{2k-1} q^{4lm} = \sum_{l,m\ge1} (-1)^{(l-1)(m-1)} l^{2k-1} q^{lm}, \tag{A.1}$$

$$2 \sum_{l,m\ge1} (2l)^{2k-1} q^{2lm} - \sum_{l,m\ge1} l^{2k-1} q^{lm} = \sum_{l,m\ge1} (-1)^l l^{2k-1} q^{lm}.$$

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